4. Consider the differential equation:

$$(*) \qquad a\frac{d^2u}{dx^2} + b\frac{du}{dx} = f(x) \text{ for } x \in (0, 2\pi),$$

where $a, b > 0$. Assume $u$ and $f$ are periodically extended to $\mathbb{R}$. Divide the interval $[0, 2\pi]$ into $n$ equal portions, where $n = 2^l$ for some $l > 10$. Let $x_j = \frac{2\pi j}{n}$ for $j = 0, 1, 2, ..., n-1$.

Let $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_{n-1}))^T$ and $\mathbf{f} = (f(x_0), f(x_1), ..., f(x_{n-1}))^T$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two $n \times n$ matrices, which are defined in such a way that:

$$(\mathcal{D}_1\mathbf{u})_j = \frac{u(x_{j+4}) - u(x_{j-4})}{8h} \quad \text{and} \quad (\mathcal{D}_2\mathbf{u})_j = \frac{u(x_{j+4}) - 2u(x_j) + u(x_{j-4})}{16h^2}.$$

for $j = 0, 1, 2, ..., n-1$ and $h = \frac{2\pi}{n}$.

(a) Using Taylor expansion, explain why $\mathcal{D}_1$ and $\mathcal{D}_2$ approximate $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ respectively. Hence, deduce that the differential equation (*) can be discretized as:

$$(**) \quad a\mathcal{D}_2\mathbf{u} + b\mathcal{D}_1\mathbf{u} = \mathbf{f}.$$

(b) Let $\mathbf{u} = \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}}$ and $\mathbf{f} = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$, where $\hat{u}_k, \hat{f}_k \in \mathbb{C}$. If $\mathbf{u}$ satisfies (**), show that

$$(a\lambda_k + b\widetilde{\lambda}_k)\hat{u}_k = \hat{f}_k \text{ for some } \lambda_k \text{ and } \widetilde{\lambda}_k,$$

for $k = 0, 1, 2, ..., n-1$. What are $\lambda_k$ and $\widetilde{\lambda}_k$? Please explain your answer with details.

(c) Let $\mathbf{u}^*$ be one of the solutions of (**). What is the general solution of (**)? Please show and explain your answer with details.

(b). $$\left(D_1 \overrightarrow{e^{ikx}}\right)_j = \frac{e^{ikx_{j+4}} - e^{ikx_{j-4}}}{8h}$$

$$= \frac{e^{ik4h} - e^{-ik4h}}{8h} e^{ikx_j}$$

$$= \frac{1}{8h}\left(\cos 4kh + i\sin 4kh - \cos 4kh + i\sin 4kh\right) e^{ikx_j}.$$

$$= \underbrace{\frac{i}{4h}\sin 4kh}_{\widetilde{\lambda}_k}\; e^{ikx_j}$$

$$(D_2 \vec{e^{ikx}})_j = \frac{e^{ikx_{j+4}} - 2e^{ikx_j} + e^{ikx_{j-4}}}{16h^2}$$

$$= \frac{1}{16h^2}\left(e^{i4kh} - 2 + e^{-i4kh}\right) e^{ikx_j}$$

$$= \frac{1}{16h^2}\left(2\cos 4kh - 2\right) e^{ikx_j}$$

$$= \frac{1}{8h^2}\left(\cos^2 2kh - \sin^2 2kh - \cos^2 2kh - \sin^2 2kh\right) e^{ikx_j}$$

$$= \underbrace{-\frac{1}{4h^2}\sin^2 2kh}_{\lambda_k} \, e^{ikx_j}$$

(c). $\vec{u}^*$ is a solution to the discretized ODE.

$$A := aD_2 + bD_1$$

discrete ODE:

$$aD_2\vec{u} + bD_1\vec{u} = f$$

$$\Leftrightarrow A\vec{u} = f.$$

If $\vec{u}^*$ is a solution,

then $\vec{u}^* + \vec{u_0}$ is also a solution, where $\vec{u}_0 \in N(A)$.

Note the null space is spanned by eigenvectors with zero eigenvalues.

eigenvalues of $A$ is given by

$$a\lambda_k + b\tilde{\lambda}_k$$

$$= \underbrace{a \cdot \frac{1}{4h}\sin^2 2kh}_{Re} + \underbrace{b\,\frac{i}{4h}\sin 4kh}_{Im}$$

$= 0$ iff $Re = 0$ and $Im = 0$.

$$0 = \frac{a}{4h}\sin^2 2kh$$

$$\Leftrightarrow \quad 0 = \sin\left(\frac{4\pi}{n}k\right), \quad k = 0, 1, \cdots, n-1$$

$$\Leftrightarrow \quad k = 0 \text{ or } \frac{n}{4} \text{ or } \frac{2}{4}n \text{ or } \frac{3}{4}n. \quad \leftarrow n = 2^{\ell}$$

$$0 = \frac{b}{4h} \sin 4kh$$

$$\Leftrightarrow \quad 0 = \sin\left(\frac{8\pi}{n} k\right), \quad k = 0, 1, \dots, n-1$$

$$k = \frac{m}{8} n, \quad m = 0, 1, \dots, 7.$$

Real Part and Imaginary Part both are zero iff

$$k = \frac{m}{4} n, \quad m = 0, 1, 2, 3.$$

$$N(A) = \operatorname{span}\left\{ \overrightarrow{e^{i0x}}, \overrightarrow{e^{i\frac{n}{4}\pi}}, \overrightarrow{e^{i\frac{n}{2}\pi}}, \overrightarrow{e^{i\frac{3}{4}n\pi}} \right\}$$

$\therefore$ a general solution is:

$$\vec{u} = \vec{u}^* + c_0 \overrightarrow{e^{i0x}} + c_1 \overrightarrow{e^{i\frac{n}{4}\pi}} + c_2 \overrightarrow{e^{i\frac{n}{2}\pi}} + c_3 \overrightarrow{e^{i\frac{3}{4}n\pi}}.$$

5. Let $A$ be a $n \times n$ complex matrix given by:

$$A = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}$$

(a) Show that $\overrightarrow{e^{ikx}}$ is an eigenvector of $A$ corresponding to an eigenvalue $\mu_k$ for $k = 0, 1, 2, ..., n-1$. What is $\mu_k$? Please show all your steps in details.

Periodically extend $c_l$:

$$c_{l+n} = c_{l-n} = c_l.$$

$$(A \overrightarrow{e^{ikx}})_j = \sum_{l=0}^{n-1} c_{l-j} e^{ikhl}$$

$$= \left( \sum_{l=0}^{n-1} c_{l-j} e^{ikh(l-j)} \right) e^{ikhj}$$

$$= \sum_{l=-j}^{n-1-j} c_l e^{ikhl} \cdot e^{ikhj}$$

$$= \left( \sum_{l=0}^{n-1-j} + \sum_{l=-j}^{-1} \right) c_l e^{ikhl} \cdot e^{ikhj}$$

$$\left(\begin{aligned} \sum_{l=-j}^{-1} c_l e^{ikhl} &= \sum_{l=-j}^{-1} c_{l+n} e^{ikh(l+n)} \\ &= \sum_{l=n-j}^{n-1} c_l e^{ikhl} \end{aligned}\right)$$

$$= \left( \sum_{l=0}^{n-1-j} + \sum_{l=n-j}^{n-1} \right) c_l e^{ikhl} \qquad e^{ikhj}$$

$$= \underline{\left( \sum_{l=0}^{n-1} c_l e^{ikhl} \right)} \, e^{ikhj}$$

Mk.

(b) Suppose $A\mathbf{x} = \mathbf{f}$, where $\mathbf{x}$ and $\mathbf{f}$ are both complex vectors in $\mathbb{C}^n$. Using (a), show that:

$$\begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{n-1} \end{pmatrix} \otimes \mathrm{DFT}(\mathbf{x}) = \mathrm{DFT}(\mathbf{f})$$

where

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_1 b_1 \\ \vdots \\ a_{n-1} b_{n-1} \end{pmatrix}.$$

Please explain your answer with details.

By iDFT on $\vec{x}$ and $\vec{f}$,

$$A\vec{x} = \vec{f}$$

$$\Rightarrow A\left(\sum_{j=0}^{n-1} \hat{x}_j \overrightarrow{e^{ijx}}\right) = \sum_{j=0}^{n-1} \hat{f}_j \overrightarrow{e^{ijx}}$$

$$\Rightarrow \sum_{j=0}^{n-1} \hat{x}_j \left(A \overrightarrow{e^{ijx}}\right) = \sum_{j=0}^{n-1} \hat{f}_j \overrightarrow{e^{ijx}}$$

$$\Rightarrow \sum_{j=0}^{n-1} \hat{x}_j \mu_j \overrightarrow{e^{ijx}} = \sum_{j=0}^{n-1} \hat{f}_j \overrightarrow{e^{ijx}}$$

Linear Independence of $\{\overrightarrow{e^{ijx}}\}$

$$\Rightarrow \hat{x}_j \mu_j = \hat{f}_j \quad \forall\, j = 0, 1, \ldots, n-1.$$

# Numerical Iterative Methods

In general, an iterative method for solving some problem $f(x) = 0$ is defined by:

$$x^{k+1} = \Phi(x^k), \quad k = 0, 1, \ldots$$

If $x^*$ is a solution, we need $x^* = \Phi(x^*)$.

Now, we consider the problem $Ax = b$,

iterative scheme $x^{k+1} = M x^k + f$,

$$A, B \in \mathbb{R}^{n \times n}, \quad b, f \in \mathbb{R}^n.$$

$$\begin{array}{rl} x^{k+1} &= M x^k + f \\ - \quad x^* &= M x^* + f \\ \hline x^{k+1} - x^* &= M(x^k - x^*) \end{array}$$

Error vector: $\overrightarrow{e^{k+1}} = M \overrightarrow{e^k}$

$$= M(M \overrightarrow{e^{k-1}})$$

$$\vdots$$

$$= M^{k+1} \overrightarrow{e^0}$$

Suppose $M \in \mathbb{R}^{n \times n}$ has $n$ linearly independent eigenvectors.

$\lambda_i$ eigenvalues, $\vec{u}_i$ eigenvectors

Suppose $|\lambda_1| = \max_i \{|\lambda_i|\}$.

$$\vec{e}_k = M^k e_0 \quad , \quad e_0 = \sum_{j=0}^{n-1} a_j \vec{u}_j$$

$$= \sum_{j=0}^{n-1} \lambda_j^k a_j \vec{u}_j .$$

$$= \lambda_1^k \left( \sum_{j=0}^{n-1} \left(\frac{\lambda_j}{\lambda_1}\right)^k a_j \vec{u}_j \right) \to 0 \text{ if } |\lambda_1| < 1 .$$

So, we have if $\underbrace{\text{abs. of } \lambda_{max}}_{\text{spectral radius } \rho} < 1$,

the iterative scheme converges

## Jacobi / Gauss - Seidel Method

For $A\vec{x} = b$,

$$A = L + D + U$$

Jacobi: $\vec{x}^{k+1} = D^{-1}(-L-U)\vec{x} + D^{-1}\vec{f}$.

$$D^{-1}(-L-U)\vec{x}^* + D^{-1}\vec{f}$$

$$= D^{-1}(-L-D-U)\vec{x}^* + D^{-1}D\vec{x}^* + D^{-1}\vec{f}$$

$$= D^{-1}(-A\vec{x}^* + \vec{f}) + \vec{x}^*$$

$$= \vec{x}^*$$

G-S: $\vec{x}^{k+1} = -(L+D)^{-1} U \vec{x}^{k} + (L+D)^{-1} \vec{f}$

$$-(L+D)^{-1} U \vec{x}^{*} + (L+D)^{-1} \vec{f}$$
$$= -(L+D)^{-1}(L+D+U)\vec{x}^{*} + (L+D)^{-1}(L+D)\vec{x}^{*}$$
$$+ (L+D)^{-1}\vec{f}$$
$$= -(L+D)^{-1}(A\vec{x}^{*} + \vec{f}) + \vec{x}^{*}$$
$$= \vec{x}^{*}.$$

## Example

$$A = \begin{bmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{bmatrix}$$

Jacobi: $M_J = \begin{bmatrix} 0, & 2/5, & -3/5 \\ 1/3, & 0, & -1/9 \\ 2/7, & -1/7, & 0 \end{bmatrix}$

G-S: $M_{GS} = \begin{bmatrix} 0, & 2/5, & -3/5 \\ 0, & 2/15, & -14/45 \\ 0, & 2/21, & -8/63 \end{bmatrix}$

To study the convergence,

we can compute eigenvalues:

$M_J$: $0.22, \; -0.11 - 0.24i, \; -0.11 + 0.24i$.

$M_{GS}$: $0, \; 0.0032 - 0.11i, \; 0.0032 + 0.11i$

Or consider Gershgorin Circle Theorem.

## Exercise

Consider

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 2 & -9 & 0 \\ 0 & -8 & 6 \end{bmatrix}$$

Find $M_J$, $M_{GS}$ and their spectral radius.

Are Jacobi or Gauss-Seidel methods converge?

Solution :

$$M_J = \begin{bmatrix} 0, & -1/4, & -1/4 \\ 2/9, & 0, & 0 \\ 0, & -4/3, & 0 \end{bmatrix}$$

eigenvalues : $0.38$, $-0.19 - 0.40i$

$-0.19 + 0.40i$

$\rho(M_J) = 0.44$

$$M_{GS} = \begin{bmatrix} 0, & -1/4, & -1/4 \\ 0, & -1/18, & -1/18 \\ 0, & 2/27, & 2/27 \end{bmatrix}$$

eigenvalue = $0, 0, 0.0185$

$\rho(M_{GS}) = 0.0185$